

# DEGENERATE PRINCIPAL SERIES OF METAPLECTIC GROUPS AND HOWE CORRESPONDENCE

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**ABSTRACT.** The main purpose of this article is to supplement the authors' results on degenerate principal series representations of real symplectic groups with the analogous results for metaplectic groups. The basic theme, as in the previous case, is that their structures are anticipated by certain natural subrepresentations constructed from Howe correspondence. This supplement is necessary as these representations play a key role in understanding the basic structure of Howe correspondence (and its complications in the archimedean case), and their global counterparts play an equally essential part in the proof of Siegel-Weil formula and its generalizations (work of Kudla-Rallis). The full results in the metaplectic case also shed light on the seeming peculiarities, when the results in the symplectic case are viewed in their isolation.

## 1. INTRODUCTION: CLASSICAL INVARIANT THEORY AND ITS TRANSCENDENTAL ANALOG

Let the complex orthogonal group  $O_m$  act on the space of complex matrices  $M_{m,n}$  by matrix multiplication on the left:

$$O_m \curvearrowright M_{m,n}.$$

The First Fundamental Theorem (FFT) of classical invariant theory asserts that the ring of  $O_m$ -invariant polynomials are generated by the fundamental invariants of degree 2:

$$P(M_{m,n})^{O_m} = \langle r_{ij} \mid 1 \leq i, j \leq n \rangle,$$

where  $r_{ij}(X) = \sum_{k=1}^m x_{ki}x_{kj}$ , for  $X = (x_{ij}) \in M_{m,n}$ .

Let  $S^2(\mathbb{C}^n)$  denote the space of complex  $n \times n$  symmetric matrices and define

$$\begin{aligned} Q : M_{m,n} &\rightarrow S^2(\mathbb{C}^n), \\ X &\mapsto X^t X. \end{aligned}$$

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Then FFT asserts that the pull-back map

$$Q^* : P(S^2(\mathbb{C}^n)) \rightarrow P(M_{m,n})^{O_m}$$

is surjective. Alternatively, we have an affine embedding:

$$\psi : M_{m,n}/O_m \hookrightarrow S^2(\mathbb{C}^n),$$

where  $M_{m,n}/O_m$  is the affine quotient of  $M_{m,n}$  by  $O_m$ . The Second Fundamental Theorem (SFT) of classical invariant theory is then a statement on the image of  $\psi$  as an affine subvariety of  $S^2(\mathbb{C}^n)$ . Note that both  $M_{m,n}$  and  $S^2(\mathbb{C}^n)$  carry natural actions of  $\mathrm{GL}(n, \mathbb{C})$ , and all maps  $(Q, Q^*, \psi)$  are  $\mathrm{GL}(n, \mathbb{C})$ -equivariant.

We now state results of Kudla-Rallis [KR1] and Lee-Zhu [LZ2], which may be viewed as transcendental analogs of FFT and SFT alluded to above.

Let

$$O(p, q) \curvearrowright M_{p+q,n}(\mathbb{R}),$$

again by matrix multiplication on the left. Denote by  $G = \widetilde{\mathrm{Sp}}(2n, \mathbb{R})$  the real metaplectic group of rank  $n$  and  $P$  its Siegel parabolic subgroup. The rest of notation will be explained in Section 3.

**Theorem 1.1.** (Kudla-Rallis) *There is a natural topological embedding with closed image:*

$$\psi_{p,q} : \mathcal{S}(M_{p+q,n}(\mathbb{R}))_{O(p,q)} \hookrightarrow I^\alpha(\sigma) := \mathrm{Ind}_P^G(\chi_\sigma^\alpha),$$

as  $G$ -representations.

**Theorem 1.2.** (Lee-Zhu) *describes the image of  $\psi_{p,q}$ .*

In [LZ2], the authors describe the image  $\Omega^{p,q}$  of  $\psi_{p,q}$  when  $p+q$  is even, in which case the representations concerned factor through the linear group  $\mathrm{Sp}(2n, \mathbb{R})$ . In fact only the subcase of  $p-q \equiv 0 \pmod{4}$  was treated in detail and the other subcase of  $p-q \equiv 2 \pmod{4}$  was left to the reader.

The aim of the current article is to complete the description of  $\Omega^{p,q}$  (without any restriction on  $p$  and  $q$ ). This is necessary and useful as the representations involved (the coinvariants) play a key role in understanding the basic structure of Howe correspondence (through the doubling method [H1, Ra, Ku]) and their global counterparts feature prominently in the proof of Siegel-Weil formula and its generalizations [KR3]. For a recent application of these results to first occurrence conjecture of Kudla-Rallis, see [SZ]. As an additional benefit, this full description enables us to organize the statements in a more coherent way so that their structures emerge clearer to the reader. The basic

idea is very simple: since both  $\Omega^{p,q}$  and  $I^\alpha(\sigma)$  are  $K$ -multiplicity free ( $K$  being a maximal compact subgroup of  $G$ ), and since the structure of  $I^\alpha(\sigma)$  is known (Section 5; [L2] for the linear group), one can identify the image  $\Omega^{p,q}$  as a  $G$ -representation by knowing its  $K$ -types. The fascinating point is that from this description, one concludes that the reducibilities of  $I^\alpha(\sigma)$  are completely accounted for by the possible embeddings of  $\Omega^{p,q}$ 's (in a precise way). We summarize this assertion in Theorem 6.1, which should be viewed as analogous to SFT (proverbially stated) as “all relations among the fundamental invariants are generated by the obvious ones”.

Here are some words on the organization of this article. In Section 2, we review the basics of Howe duality correspondence. In Section 3, we introduce the space of coinvariants, as a special case of Howe's construction of maximal quotients, and its embedding into the degenerate principal series  $I^\alpha(\sigma)$  of the metaplectic group. In Section 4, we describe the transition coefficients of  $I^\alpha(\sigma)$  and give its immediate consequences for irreducibility and complementary series. In Section 5, we give the detailed structure of  $I^\alpha(\sigma)$  at points of reducibility, which are again arrived by analyzing the transition coefficients. In Section 6, we complete our description of the image of  $\psi_{p,q}$  in  $I^\alpha(\sigma)$ . As proofs of (any new) results follow a similar line as those of [L2] or [LZ2], we shall omit them.

Finally we mention the following works which are closely related to the theme of this article: [KR1, LZ1, LZ2, LZ3, Ya] (for real groups) and [KR2, KS, Ya] (for  $p$ -adic groups).

## 2. HOWE DUALITY CORRESPONDENCE

In this section, we review briefly Howe's theory of dual pair correspondence, for the case at hand.

Let  $H = \mathrm{O}(p, q)$ , which acts on  $M_{p+q,n}(\mathbb{R})$ . We have the dualised action of  $H$  on  $\mathcal{Y} = L^2(M_{p+q,n}(\mathbb{R}))$ , the space of square integrable functions on  $M_{p+q,n}(\mathbb{R})$ . Extending the action of  $H$ , there is a (unitary) representation of a metaplectic group  $\widetilde{\mathrm{Sp}}(2N, \mathbb{R})$  ( $N = (p+q)n$ ) on  $\mathcal{Y}$ , called the Schrodinger model of an oscillator representation. Here and after, for any real symplectic group  $\mathrm{Sp}(2N, \mathbb{R})$ ,  $\widetilde{\mathrm{Sp}}(2N, \mathbb{R})$  denotes its metaplectic two fold cover. This representation depends on a choice of a non-trivial unitary character of  $\mathbb{R}$ , which we fix once for all. Denote the corresponding oscillator representation by  $\omega$ .

The main feature of this set-up is the following: there is a reductive dual pair

$$(H, G) = (\mathrm{O}(p, q), \mathrm{Sp}(2n, \mathbb{R})) \subseteq \mathrm{Sp}(2N, \mathbb{R}),$$

namely a pair of reductive subgroups in  $\mathrm{Sp}(2N, \mathbb{R})$  which are mutual centralizers.

Let  $\omega^\infty$  be the smooth representation of  $\omega$  realized in  $\mathcal{Y}^\infty$ . For the case at hand,  $\mathcal{Y}^\infty = \mathcal{S}(M_{p+q,n}(\mathbb{R}))$ , the Schwartz space of rapidly decreasing functions on  $M_{p+q,n}(\mathbb{R})$ .

For any subgroup  $E$  of  $\widetilde{\mathrm{Sp}}(2N, \mathbb{R})$ , denote by  $\widetilde{E}$  the preimage of  $E$  under the covering map  $\widetilde{\mathrm{Sp}}(2N, \mathbb{R}) \rightarrow \mathrm{Sp}(2N, \mathbb{R})$ . If  $E$  is also reductive, denote by  $\mathcal{R}(\widetilde{E}, \omega)$  the set of infinitesimal equivalent classes of irreducible admissible representations of  $\widetilde{E}$  which are realizable as quotients by  $\widetilde{E}$ -invariant closed subspaces of  $\mathcal{Y}^\infty$ .

**Howe Duality Theorem** ([H2]):  $\mathcal{R}(\widetilde{H} \cdot \widetilde{G}, \omega)$  is the graph of a bijection between  $\mathcal{R}(\widetilde{H}, \omega)$  and  $\mathcal{R}(\widetilde{G}, \omega)$ .

This is the Howe quotient correspondence. To be more concrete, let  $\rho$  be an irreducible admissible representation of  $\widetilde{H}$ , which is realizable as a quotient by a  $\widetilde{H}$ -invariant closed subspace of  $\mathcal{Y}^\infty$ . Define  $\Omega(\rho)$  to be the maximal quotient of  $\mathcal{Y}^\infty$  on which  $\widetilde{H}$  acts by a representation of class  $\rho$ . We have (isomorphism class to mean infinitesimal equivalent class):

$$\Omega(\rho) \cong \rho \otimes \Theta(\rho)$$

where  $\Theta(\rho)$  is a  $\widetilde{G}$ -module.

Howe duality theorem asserts that  $\Theta(\rho)$  is a finitely generated admissible quasisimple representation of  $\widetilde{G}$ , and has a unique irreducible  $\widetilde{G}$ -quotient, denoted by  $\theta(\rho)$ :

$$\Theta(\rho) \twoheadrightarrow \theta(\rho).$$

The correspondence

$$\rho \mapsto \theta(\rho)$$

is the Howe quotient correspondence.

### 3. THE COINVARIANTS AND THE EMBEDDING

For the dual pair  $(H, G) = (\mathrm{O}(p, q), \mathrm{Sp}(2n, \mathbb{R}))$ , let

$$\begin{aligned} \Omega^{p,q} &= \text{Howe's maximal quotient corresponding to} \\ &\quad \text{the trivial representation } \mathbb{1} \text{ of } \mathrm{O}(p, q) \\ &= \mathcal{S}(M_{p+q,n}(\mathbb{R}))_{\mathrm{O}(p,q)} \quad (\text{the space of coinvariants}). \end{aligned}$$

This is a representation of  $\widetilde{G}$ .

**Remark:** the continuous dual of  $\Omega^{p,q}$  is  $\mathcal{S}^*(M_{p+q,n}(\mathbb{R}))^{\mathrm{O}(p,q)}$ , the space of  $\mathrm{O}(p,q)$ -invariant tempered distributions on  $M_{p+q,n}(\mathbb{R})$ . This is investigated in [KR1] and [Zh].

From now on, we shift notation and will denote  $G = \widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ . The case of  $n = 1$  differs from the general case slightly, but since it is straightforward, we omit it and will assume that  $n \geq 2$  throughout this paper. We shall identify  $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$  as a set with

$$\mathrm{Sp}(2n, \mathbb{R}) \times \mathbb{Z}_2 = \{(g, \varepsilon) : g \in \mathrm{Sp}(2n, \mathbb{R}), \varepsilon = \pm 1\}.$$

For  $a \in \mathrm{GL}(n, \mathbb{R})$  and  $b \in M_n(\mathbb{R})$  such that  $b = b^t$ , we let

$$\begin{aligned} m_a &= \begin{pmatrix} a & 0 \\ 0 & (a^{-1})^t \end{pmatrix}, \\ n_b &= \begin{pmatrix} I_n & b \\ 0 & I_n \end{pmatrix}. \end{aligned}$$

Let

$$M = \{(m_a, \varepsilon) : a \in \mathrm{GL}(n, \mathbb{R}), \varepsilon = \pm 1\}$$

and

$$N = \{(n_b, 1) : b \in M_n(\mathbb{R}), b = b^t\}.$$

Then  $P = MN$  is a maximal parabolic subgroup of  $G$ , called the Siegel parabolic.

Let  $\chi : M \rightarrow \mathbb{C}^\times$  be given by

$$\chi(m_a, \varepsilon) = \varepsilon \cdot \begin{cases} i & \text{if } \det a < 0, \\ 1 & \text{if } \det a > 0. \end{cases}$$

This is a character of  $M$  and it is of order 4. For  $\alpha = 0, 1, 2, 3$  and  $\sigma \in \mathbb{C}$ , let  $\chi_\sigma^\alpha$  be the character of  $P$  given by

$$\chi_\sigma^\alpha[(m_a, \varepsilon)(n_b, 1)] = |\det a|^\sigma \chi(m_a, \varepsilon)^\alpha. \quad (3.1)$$

Let  $I^\alpha(\sigma)$  be the normalized induced representation:

$$I^\alpha(\sigma) = \mathrm{Ind}_P^G \chi_\sigma^\alpha.$$

The representation space of  $I^\alpha(\sigma)$  is

$$\{f \in C^\infty(G) : f(pg) = \delta^{\frac{1}{2}}(p) \chi_\sigma^\alpha(p) f(g), \forall g \in G, p \in P\},$$

and  $G$  acts by right translation:

$$g \cdot f(h) = f(hg), \quad (g, h \in G).$$

Here  $\delta$  denotes the modular function of  $P$ , and is given by

$$\delta[(m_a, \varepsilon)(n_b, 1)] = |\det a|^{2\rho_n}$$

and

$$\rho_n = \frac{n+1}{2}.$$

When  $\alpha = 0$  or  $2$ , the representation  $I^\alpha(\sigma)$  descends to a representation of the linear group  $\mathrm{Sp}(2n, \mathbb{R})$ .

Define the map  $\psi_{p,q} : \mathcal{S}(M_{p+q,n}(\mathbb{R})) \mapsto C^\infty(G)$  by

$$\psi_{p,q}(f)(g) = (\omega(\tilde{g})f)(0), \quad f \in \mathcal{S}(M_{p+q,n}(\mathbb{R})), \quad g \in G.$$

From the well-known formula of the oscillator representation  $\omega$ , we see that

$$\psi_{p,q} : \mathcal{S}(M_{p+q,n}(\mathbb{R})) \mapsto I^\alpha(\sigma),$$

where

$$\sigma = \frac{p+q}{2} - \rho_n, \quad \text{and} \quad \alpha \equiv p - q \pmod{4}. \quad (3.2)$$

The following is the fundamental result of Kudla-Rallis.

**Theorem 3.3.** ([KR1]) *The map  $\psi_{p,q}$  induces a topological embedding with closed image:*

$$\Omega^{p,q} \hookrightarrow I^\alpha(\sigma)$$

**Remark:** Analogous results hold for other classical groups. See [Zh].

#### 4. DEGENERATE PRINCIPAL SERIES: THE TRANSITION COEFFICIENTS

Fix a maximal compact subgroup  $K_1 \simeq \mathrm{U}(n)$  of  $\mathrm{Sp}(2n, \mathbb{R})$  and let  $K$  be the inverse image of  $K_1$  in  $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ , thus a maximal compact subgroup of  $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ . The Lie algebra of  $\mathrm{Sp}(2n, \mathbb{R})$  has a Cartan decomposition

$$\mathfrak{sp}(2n, \mathbb{R}) = \mathfrak{k} \oplus \mathfrak{p}$$

where  $\mathfrak{k}$  is the Lie algebra of  $K_1$  and is isomorphic to  $\mathfrak{u}(n)$ .

Let

$$\Lambda_n^+ = \{(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n : \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n\},$$

and

$$\mathbf{1} = (1, \dots, 1) \in \Lambda_n^+.$$

Also let  $e_j = (\overbrace{0, \dots, 0}^j, 1, 0, \dots, 0)$ , for  $1 \leq j \leq n$ .

We have the  $K$ -type decomposition:

$$I^\alpha(\sigma)|_K = \bigoplus_{\lambda \in \Lambda_n^+} V_{2\lambda + \frac{\alpha}{2}\mathbf{1}}, \quad (4.1)$$

where  $V_{2\lambda + \frac{\alpha}{2}\mathbf{1}}$  is an irreducible  $K$ -module with highest weight  $2\lambda + \frac{\alpha}{2}\mathbf{1}$ , for  $\lambda \in \Lambda_n^+$ .

For  $\mu = 2\lambda + \frac{\alpha}{2}\mathbf{1}$ , let  $V_\mu$  be a  $K$ -type in  $I^\alpha(\sigma)$  and fix  $\gamma_\mu$  to the unique (up to a multiple)  $K$ -highest weight vector in  $V_\mu$ . We consider the tensor product  $\mathfrak{p}_\mathbb{C} \otimes V_\mu$  of  $\mathfrak{k}_\mathbb{C}$  modules, and the  $\mathfrak{k}_\mathbb{C}$  map

$$\begin{aligned} m : \mathfrak{p}_\mathbb{C} \otimes V_\mu &\longrightarrow I^\alpha(\sigma)_K \\ m(p \otimes v) &= p \cdot v. \end{aligned}$$

For each  $1 \leq j \leq n$ , there exists an element  $X_j$  in  $\mathcal{U}(\mathfrak{sp}(2n, \mathbb{C}))$  (the universal enveloping algebra of the complexified Lie algebra) with the property that  $X_j(\gamma_\mu)$  is the image of the unique  $\mathfrak{k}_\mathbb{C}$  highest weight vector of weight  $\mu + 2e_j$  in  $\mathfrak{p}_\mathbb{C} \otimes V_\mu$ , valid for all  $\mu$ . C.f. [L2, Section 3]. Thus  $X_j(\gamma_\mu)$  is a multiple of the unique  $\mathfrak{k}_\mathbb{C}$  highest weight vector  $\gamma_{\mu+2e_j}$  in  $V_{\mu+2e_j}$ . We call this multiple the transition coefficient from  $V_\mu$  to  $V_{\mu+2e_j}$ . Likewise there is a transition coefficient from  $V_\mu$  to  $V_{\mu-2e_j}$ . Note that the transition coefficients depend on the choice of  $\gamma_\mu$ 's.

By explicitly constructing an appropriate choice of  $\mathfrak{k}_\mathbb{C}$  highest weight vector  $\gamma_\mu$  for all  $\mu$  (c.f. [L2, Section 3]), the transition coefficients  $A_j^+(\mu)$  from  $V_\mu$  to  $V_{\mu+2e_j}$ , and the transition coefficients  $A_j^-(\mu)$  from  $V_\mu$  to  $V_{\mu-2e_j}$ , are computed to be

$$A_j^+(\mu) = B_j^+ - 2\lambda_j, \quad A_j^-(\mu) = 2\lambda_j - B_j^-, \quad (4.2)$$

where

$$B_j^+ = -\sigma - \rho_n - \frac{\alpha}{2} + j - 1, \quad B_j^- = \sigma - \rho_n - \frac{\alpha}{2} + j + 1. \quad (4.3)$$

**Corollary 4.4.** ([KR1])  *$I^\alpha(\sigma)$  is irreducible if and only if*

$$\sigma + \rho_n + \frac{\alpha}{2} \notin \mathbb{Z}.$$

The explicit form of transition coefficients also allows us to determine the complementary series. We will follow the method used in [L2]. Each  $K$ -type  $V_{2\lambda + \frac{\alpha}{2}\mathbf{1}}$  of  $I^\alpha(\sigma)$  has a  $K$ -invariant inner product given by

$$\langle f_1, f_2 \rangle_\lambda = \int_K f_1(k) \overline{f_2(k)} dk.$$

Since  $V_{2\lambda + \frac{\alpha}{2}\mathbf{1}}$  is an irreducible  $K$ -module, any  $K$ -invariant inner product on  $V_{2\lambda + \frac{\alpha}{2}\mathbf{1}}$  is a multiple of  $\langle \cdot, \cdot \rangle_\lambda$ . Thus if  $\langle \cdot, \cdot \rangle$  is a  $\widetilde{\text{Sp}}(2n, \mathbb{R})$ -invariant inner product on  $I^\alpha(\sigma)$  then there exists positive constants  $\{c_\lambda\}_{\lambda \in \Lambda_n^+}$  such that

$$\langle f_1, f_2 \rangle = c_\lambda \langle f_1, f_2 \rangle_\lambda, \quad \forall f_1, f_2 \in V_{2\lambda + \frac{\alpha}{2}\mathbf{1}}.$$

Since the  $K$ -types of  $I^\alpha(\sigma)$  are mutually orthogonal with respect  $\langle \cdot, \cdot \rangle$ ,  $\langle \cdot, \cdot \rangle$  is completely determined by the constants  $\{c_\lambda\}$ . Using similar arguments as the  $U(n, n)$  case (see [L1, section 9]), we obtain the following:

**Lemma 4.5.** *The inner product on  $I^\alpha(\sigma)$  defined by the constants  $\{c_\lambda\}_{\lambda \in \Lambda_n^+}$  is  $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ -invariant if and only if*

$$(-\sigma - \rho_n - \frac{\alpha}{2} - 2\lambda_j + j - 1)c_{\lambda+e_j} + (-\bar{\sigma} + \rho_n + \frac{\alpha}{2} + 2\lambda_j - j + 1)c_\lambda = 0$$

for all  $\lambda \in \Lambda_n^+$  and all  $1 \leq j \leq n$ .

We let

$$N_{\lambda,j} = \frac{-\sigma - \rho_n - \frac{\alpha}{2} - 2\lambda_j + j - 1}{-\bar{\sigma} + \rho_n + \frac{\alpha}{2} + 2\lambda_j - j + 1} = -\frac{c_\lambda}{c_{\lambda+e_j}}.$$

Then  $I^\alpha(\sigma)$  is unitarizable if and only if  $N_{\lambda,j} < 0$  for all  $\lambda \in \Lambda_n^+$  and for all  $j$ .

Note that

$$N_{\lambda,j} = \frac{-\sigma - (\rho_n + \frac{\alpha}{2} + 2\lambda_j - j + 1)}{-\bar{\sigma} + (\rho_n + \frac{\alpha}{2} + 2\lambda_j - j + 1)}.$$

We write  $\xi = \rho_n + \frac{\alpha}{2} + 2\lambda_j - j + 1$ . Then  $N_{\lambda,j} = (-\sigma - \xi)/(-\bar{\sigma} + \xi)$ . Thus  $N_{\lambda,j}$  is real for all  $\lambda$  and for  $j$  if and only if either  $\mathrm{Re}(\sigma) = 0$  or  $\sigma$  is real. The case  $\mathrm{Re}(\sigma) = 0$  corresponds to the unitary axis. If  $\sigma$  is real, then

$$N_{\lambda,j} = \frac{-\sigma - \xi}{-\bar{\sigma} + \xi} < 0 \iff |\sigma| < |\xi|.$$

The minimum value of  $|\xi|$  is  $\frac{1}{2}$  if  $n + \alpha$  is even, and 0 if  $n + \alpha$  is odd. This leads to the following

**Theorem 4.6.** *If  $n + \alpha$  is even, then  $I^\alpha(\sigma)$  is unitarizable for  $|\sigma| < \frac{1}{2}$ .*

## 5. SUBQUOTIENTS OF $I^\alpha(\sigma)$

In this section, we shall give a detailed description of the module structure of  $I^\alpha(\sigma)$  when it is reducible. We shall describe all the irreducible constituents of  $I^\alpha(\sigma)$  and determine which of them are unitarizable, i.e., possess a  $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ -invariant positive definite inner product. We also describe the socle series and module diagram of  $I^\alpha(\sigma)$ .

Let

$$\tilde{\sigma} = \sigma + \rho_n + \frac{\alpha}{2} = \sigma + \frac{n + 1 + \alpha}{2}. \quad (5.1)$$



By Corollary 4.4,  $I^\alpha(\sigma)$  is irreducible if and only if  $\tilde{\sigma} \notin \mathbb{Z}$ . Thus we shall assume that  $\tilde{\sigma} \in \mathbb{Z}$  throughout this section. Let  $\mu = 2\lambda + (\alpha/2)\mathbf{1}_n$ . Then by the results on transition coefficients in Section 4,

$$V_\mu \rightarrow V_{\mu+2e_j} \iff 2\lambda_j \neq B_j^+ \quad \text{and} \quad V_\mu \rightarrow V_{\mu-2e_j} \iff 2\lambda_j \neq B_j^-$$

where

$$B_j^+ = B_j^+(\alpha, \sigma) = -\sigma - \frac{n + \alpha + 1}{2} + j - 1 = -\tilde{\sigma} + j - 1$$

and

$$B_j^- = B_j^-(\alpha, \sigma) = \sigma - \frac{n + \alpha + 1}{2} + j + 1 = \tilde{\sigma} - (n + \alpha) + j.$$

Our method to determine module structure is as follows:

- (1) Represent  $V_\mu$  by the point  $2\lambda \in \mathbb{R}^n$ , where  $(x_1, \dots, x_n)$  is the standard coordinates for  $\mathbb{R}^n$ .
- (2) Barrier to block  $j$ -th rightward movement  $V_\mu \rightarrow V_{\mu+2e_j}$  at  $\ell_j^+ : x_j = B_j^+$ .
- (3) Barrier to block  $j$ -th leftward movement  $V_\mu \rightarrow V_{\mu-2e_j}$  at  $\ell_j^- : x_j = B_j^-$ .

The barrier  $\ell_j^+ : x_j = B_j^+$  is effective if and only if it cuts at an “even” point, i.e.  $B_j^+ \in 2\mathbb{Z}$ . The parity of  $B_j^+$  depends on the parity of  $\tilde{\sigma} = \sigma + \frac{n+\alpha+1}{2}$  and the parity of  $j$ . Similarly for the barrier  $\ell_j^-$ , but the parity of  $B_j^-$  also depends on the parity of  $n + \alpha$ . So our analysis will be divided into 4 cases.

$\tilde{\sigma} \setminus n + \alpha$	odd	even
odd	Case 1a	Case 2a
even	Case 1b	Case 2b

We also define the “gap” between  $\ell_j^+$  and  $\ell_j^-$  by

$$\text{gap} = B_j^+ - B_j^- = -2\sigma - 2. \quad (5.2)$$

### 5.1. Subquotients of $I^\alpha(\sigma)$ : $n + \alpha$ odd.

#### Case 1: $n + \alpha$ odd.

Since  $\tilde{\sigma} \in \mathbb{Z}$  and  $\frac{n+\alpha+1}{2} \in \mathbb{Z}$ , we have  $\sigma \in \mathbb{Z}$ . It follows from this and equation (5.2) that the gap  $B_j^+ - B_j^-$  is even, i.e.  $B_j^+ \equiv B_j^- \pmod{2}$ .

This means that for each  $j$ , either both barriers  $\ell_j^+$  and  $\ell_j^-$  are effective, or both are not effective.

We consider two subcases:

*Case 1a:*  $\tilde{\sigma} = \sigma + \frac{n+\alpha+1}{2}$  is odd      *Case 1b:*  $\tilde{\sigma} = \sigma + \frac{n+\alpha+1}{2}$  is even.

For Case 1a, We have

$$B_j^+ = -(\sigma + \frac{n+\alpha+1}{2}) + j - 1 \in 2\mathbb{Z} \iff j \text{ is even.}$$

Hence the effective barriers are  $\ell_{2i}^+$  and  $\ell_{2i}^-$  for all  $1 \leq i \leq n_0/2$ , where

$$n_0 = 2 \left\lfloor \frac{n}{2} \right\rfloor = \text{largest even integer less than or equal to } n. \quad (5.1.1)$$

But for Case 1b, We have

$$B_j^+ = -(\sigma + \frac{n+\alpha+1}{2}) + j - 1 \in 2\mathbb{Z} \iff j \text{ is odd.}$$

So the effective barriers are  $\ell_{2i-1}^+$  and  $\ell_{2i-1}^-$  for all  $1 \leq i \leq (n_1 + 1)/2$ , where

$$n_1 = 2 \left\lceil \frac{n+1}{2} \right\rceil - 1 = \text{largest odd integer less than or equal to } n. \quad (5.1.2)$$

**Case 1a:**  $\tilde{\sigma} = \sigma + \frac{n+\alpha+1}{2}$  is odd

Let  $i$  and  $j$  be such that  $0 \leq i + j \leq n_0/2$ . We will define  $R_{ij}(n, \sigma, \alpha)$  as follows:

(i) For  $\sigma \leq -1$ , let  $R_{ij}(n, \sigma, \alpha)$  be the set of  $2\lambda$  such that

$$2\lambda_{2i} \geq B_{2i+2}^+ \geq 2\lambda_{2i+2} \quad (5.1.3)$$

$$2\lambda_{n_0-2j} \geq B_{n_0-2j}^- \geq 2\lambda_{n_0-2j+2} \quad (5.1.4)$$

where

$$B_{2i+2}^+ = -\sigma - \frac{n+\alpha+1}{2} + 2i + 1 = -\sigma - \frac{n+\alpha-1}{2} + 2i,$$

$$B_{n_0-2j}^- = \sigma - \frac{n+\alpha+1}{2} + n_0 - 2j + 1 = \sigma - \frac{n+\alpha-1}{2} + n_0 - 2j.$$

(ii) For  $\sigma \geq 0$ , let  $R_{ij}(n, \sigma, \alpha)$  be the set of  $2\lambda$  such that

$$2\lambda_{2i} \geq B_{2i}^- \geq 2\lambda_{2i+2} \quad (5.1.5)$$

$$2\lambda_{n_0-2j} \geq B_{n_0-2j+2}^+ \geq 2\lambda_{n_0-2j+2} \quad (5.1.6)$$

where

$$B_{2i}^- = \sigma - \frac{n + \alpha + 1}{2} + 2i + 1 = \sigma - \frac{n + \alpha - 1}{2} + 2i,$$

$$B_{n_0-2j+2}^+ = -\sigma - \frac{n + \alpha + 1}{2} + n_0 - 2j + 1 = -\sigma - \frac{n + \alpha - 1}{2} + n_0 - 2j.$$

**Remarks:**

(a) When  $\sigma = 0$  and  $i + j = n_0/2$ , the two conditions (5.1.5) and (5.1.6) coincide. More precisely,

$$R_{ij}(n, 0, \alpha) = \{2\lambda : 2\lambda_{2i} \geq B_{2i}^- \geq 2\lambda_{2i+2}\}.$$

(b) The set  $R_{ij}(n, \sigma, \alpha)$  defined above may be empty. When it is nonempty, it can be identified with the direct sum of all the  $K$  representations  $V_{2\lambda + \frac{\alpha}{2}\mathbf{1}}$  such that  $2\lambda \in R_{ij}(n, \sigma, \alpha)$ .

**Theorem 5.1.7.** (Case 1a) *Assume that  $n + \alpha$  is odd,  $\sigma$  is an integer and  $\tilde{\sigma} = \sigma + \frac{n+\alpha+1}{2}$  is odd.*

(a) *If  $\sigma \leq -1$ , then*

$$I^\alpha(\sigma) = \bigoplus \left\{ R_{ij}(n, \sigma, \alpha) : r_1 \leq i + j \leq \frac{n_0}{2} \right\},$$

where

$$r_1 = \max \left( \frac{n_0}{2} + \sigma, 0 \right).$$

*In this case, the module diagram of  $I^\alpha(\sigma)$  can be obtained from Figure 1 by removing those  $R_{ab}(n, \sigma, \alpha)$  which are empty. In particular, the socle series of  $I^\alpha(\sigma)$  is given by*

$$\text{Soc}^l(I^\alpha(\sigma)) = \begin{cases} \bigoplus_{r_1 \leq i+j \leq r_1+l-1} R_{ij}(n, \sigma, \alpha) & 1 \leq l \leq \frac{n_0}{2} - r_1, \\ I^\alpha(\sigma) & l \geq \frac{n_0}{2} - r_1 + 1. \end{cases}$$

*An irreducible constituent  $R_{ij}(n, \sigma, \alpha)$  of  $I^\alpha(\sigma)$  is unitarizable if and only if  $-\frac{n_0}{2} \leq \sigma \leq -1$  and  $i + j = r_1$ .*

(b) If  $\sigma = 0$ , then

$$I^\alpha(0) = \bigoplus_{i+j=n_0/2} R_{ij}(n, 0, \alpha)$$

is a direct sum of irreducible unitary submodules.

(c) If  $\sigma \geq 1$ , then

$$I^\alpha(\sigma) = \bigoplus \left\{ R_{ij}(n, \sigma, \alpha) : r_2 \leq i + j \leq \frac{n_0}{2} \right\},$$

where

$$r_2 = \max \left( \frac{n_0}{2} - \sigma, 0 \right).$$

In this case, the module diagram of  $I^\alpha(\sigma)$  can be obtained from Figure 2 by removing those  $R_{ab}(n, \sigma, \alpha)$  which are empty. In particular, the socle series of  $I^\alpha(\sigma)$  is given by

$$\text{Soc}^l(I^\alpha(\sigma)) = \begin{cases} \bigoplus_{\frac{n_0}{2}-l+1 \leq i+j \leq \frac{n_0}{2}} R_{ij}(n, \sigma, \alpha) & 1 \leq l \leq \frac{n_0}{2} - r_2, \\ I^\alpha(\sigma) & l \geq \frac{n_0}{2} - r_2 + 1. \end{cases}$$

An irreducible constituent  $R_{ij}(n, \sigma, \alpha)$  of  $I^\alpha(\sigma)$  is unitarizable if and only if  $1 \leq \sigma \leq \frac{n_0}{2}$  and  $i + j = r_2$ .

**Case 1b:**  $\tilde{\sigma} = \sigma + \frac{n+\alpha+1}{2}$  is even

Let  $i$  and  $j$  be such that  $0 \leq i + j \leq (n_1 + 1)/2$ . We will define  $R_{ij}(n, \sigma, \alpha)$  as follows:

(i) For  $\sigma \leq -1$ , let  $R_{ij}(n, \sigma, \alpha)$  be the set of  $2\lambda$  such that

$$2\lambda_{2i-1} \geq B_{2i+1}^+ \geq 2\lambda_{2i+1} \quad (5.1.8)$$

$$2\lambda_{n_1-2j} \geq B_{n_1-2j}^- \geq 2\lambda_{n_1-2j+2} \quad (5.1.9)$$

where

$$B_{2i+1}^+ = -\sigma - \frac{n + \alpha + 1}{2} + 2i,$$

$$B_{n_1-2j}^- = \sigma - \frac{n + \alpha + 1}{2} + n_1 - 2j + 1 = \sigma - \frac{n + \alpha - 1}{2} + n_1 - 2j.$$

(ii) For  $\sigma \geq 0$ , let  $R_{ij}(n, \sigma, \alpha)$  be the set of  $2\lambda$  such that

$$2\lambda_{2i-1} \geq B_{2i-1}^- \geq 2\lambda_{2i+1} \quad (5.1.10)$$

$$2\lambda_{n_1-2j} \geq B_{n_1-2j+2}^+ \geq 2\lambda_{n_1-2j+2} \quad (5.1.11)$$

where

$$B_{2i-1}^- = \sigma - \frac{n + \alpha + 1}{2} + 2i,$$

$$B_{n_1-2j+2}^+ = -\sigma - \frac{n + \alpha + 1}{2} + n_1 - 2j + 1 = -\sigma - \frac{n + \alpha - 1}{2} + n_1 - 2j.$$

**Remark:** When  $\sigma = 0$  and  $i + j = (n_1 + 1)/2$ , the two conditions (5.1.8) and (5.1.9) coincide, that is,  $2\lambda_{n_1-2j} = 2\lambda_{(2i+2j-1)-2j} = 2\lambda_{2i-1}$ ,

$$B_{n_1-2j+2}^- = -\frac{n + \alpha - 1}{2} + (2i + 2j - 1) - 2j = -\frac{n + \alpha + 1}{2} + 2i = B_{2i-1}^-,$$

and  $2\lambda_{n_1-2j+2} = 2\lambda_{(2i+2j-1)-2j+2} = 2\lambda_{2j+1}$ . Precisely,

$$R_{ij}(n, 0, \alpha) = \{2\lambda : 2\lambda_{2i-1} \geq B_{2i-1}^- \geq 2\lambda_{2i+1}\}.$$

**Theorem 5.1.12.** (Case 1b) Assume that  $n + \alpha$  is odd,  $\sigma$  is an integer and  $\tilde{\sigma} = \sigma + \frac{n+\alpha+1}{2}$  is even.

(a) If  $\sigma \leq -1$ , then

$$I^\alpha(\sigma) = \bigoplus \left\{ R_{ij}(n, \sigma, \alpha) : r_1 \leq i + j \leq \frac{n_1 + 1}{2} \right\},$$

where

$$r_1 = \max \left( \frac{n_1 + 1}{2} + \sigma, 0 \right).$$

In this case, the module diagram of  $I^\alpha(\sigma)$  can be obtained from Figure 1 by removing those  $R_{ab}(n, \sigma, \alpha)$  which are empty. In particular, the socle series of  $I^\alpha(\sigma)$  is given by

$$\text{Soc}^l(I^\alpha(\sigma)) = \begin{cases} \bigoplus_{r_1 \leq i+j \leq r_1+l-1} R_{ij}(n, \sigma, \alpha) & 1 \leq l \leq \frac{n_1+1}{2} - r_1, \\ I^\alpha(\sigma) & l \geq \frac{n_1+1}{2} - r_1 + 1. \end{cases}$$

An irreducible constituent  $R_{ij}(n, \sigma, \alpha)$  of  $I^\alpha(\sigma)$  is unitarizable if and only if

- (i)  $-\frac{n_1+1}{2} \leq \sigma \leq -1$  and  $i + j = r_1$ ; or
- (ii)  $n$  is odd,  $\alpha \in \{0, 2\}$  and  $(i, j) \in \left\{ \left( \frac{n+1}{2}, 0 \right), \left( 0, \frac{n+1}{2} \right) \right\}$ .

(b) If  $\sigma = 0$ , then

$$I^\alpha(0) = \bigoplus_{i+j=\frac{n_1+1}{2}} R_{ij}(n, 0, \alpha)$$

is a direct sum of irreducible unitary submodules.

(c) If  $\sigma \geq 1$ , then

$$I^\alpha(\sigma) = \bigoplus \left\{ R_{ij}(n, \sigma, \alpha) : r_2 \leq i+j \leq \frac{n_1+1}{2} \right\},$$

where

$$r_2 = \max \left( \frac{n_1+1}{2} - \sigma, 0 \right).$$

In this case, the module diagram of  $I^\alpha(\sigma)$  can be obtained from Figure 2 by removing those  $R_{ab}(n, \sigma, \alpha)$  which are empty. In particular, the socle series of  $I^\alpha(\sigma)$  is given by

$$\text{Soc}^l(I^\alpha(\sigma)) = \begin{cases} \bigoplus_{\frac{n_1+1}{2}-l+1 \leq i+j \leq \frac{n_1+1}{2}} R_{ij}(n, \sigma, \alpha) & 1 \leq l \leq \frac{n_1+1}{2} - r_2, \\ I^\alpha(\sigma) & l \geq \frac{n_1+1}{2} - r_2 + 1. \end{cases}$$

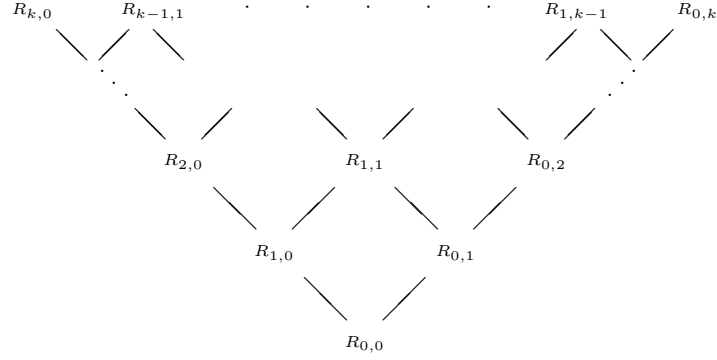
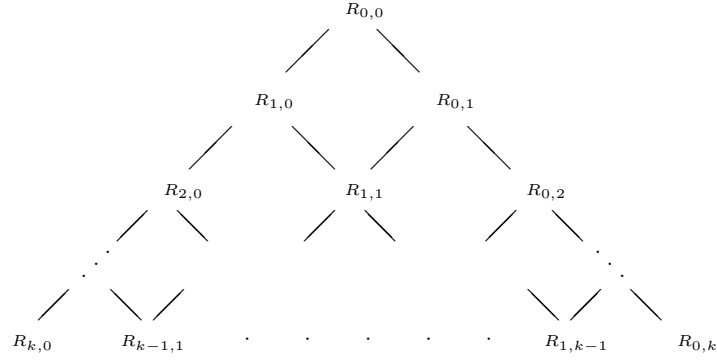
An irreducible constituent  $R_{ij}(n, \sigma, \alpha)$  of  $I^\alpha(\sigma)$  is unitarizable if and only if

- (i)  $1 \leq \sigma \leq \frac{n_1+1}{2}$  and  $i+j = r_2$ ; or
- (ii)  $n$  is odd,  $\alpha \in \{0, 2\}$  and  $(i, j) \in \left\{ \left( \frac{n+1}{2}, 0 \right), \left( 0, \frac{n+1}{2} \right) \right\}$ .

The module diagram of  $I^\alpha(\sigma)$  for  $\sigma \leq -1$  and  $\sigma \geq 1$  are given in Figure 1 and Figure 2 below. Here

$$k = \begin{cases} \frac{n_0}{2} = \left[ \frac{n}{2} \right] & \text{if } \tilde{\sigma} \text{ is odd,} \\ \frac{n_1+1}{2} = \left[ \frac{n+1}{2} \right] & \text{if } \tilde{\sigma} \text{ is even.} \end{cases} \quad (5.1.13)$$

and we write a constituent  $R_{ij}(n, \sigma, \alpha)$  simply as  $R_{ij}$ .

Figure 1: Module diagram for  $I^\alpha(\sigma)$  ( $\sigma \leq -1$ )Figure 2: Module diagram for  $I^\alpha(\sigma)$  ( $\sigma \geq 1$ )

## 5.2. Subquotients of $I^\alpha(\sigma)$ : $n + \alpha$ even.

### Case 2: $n + \alpha$ even.

Recall that  $\tilde{\sigma} = \sigma + \frac{n+\alpha+1}{2} \in \mathbb{Z}$ . So in this case,  $\sigma + \frac{1}{2} \in \mathbb{Z}$ , and consequently the gap between the barriers  $\ell_j^+$  and  $\ell_j^-$  given by

$$\text{gap} = -2\sigma - 2$$

is odd. It follows that along each coordinate axis, exactly one barrier is effective. More precisely, for each  $1 \leq j \leq n$ , either  $\ell_j^+$  is effective or  $\ell_j^-$  is effective, but not both. Again, we consider two subcases:

Case 2a:  $\tilde{\sigma} = \sigma + \frac{n+\alpha+1}{2}$  is odd

Case 2b:  $\tilde{\sigma} = \sigma + \frac{n+\alpha+1}{2}$  is even.

For Case 2a,  $\ell_j^+$  is effective when  $j$  is even, and  $\ell_j^-$  is effective when  $j$  is odd. Case 2b is the other way round.

**Case 2a:**  $\tilde{\sigma} = \sigma + \frac{n+\alpha+1}{2}$  is odd

Since exactly one barrier is effective along each coordinate axis, the barrier partitions the  $K$ -types into 2 subsets. For  $1 \leq r \leq \frac{n_1+1}{2} + 1$ , let

$$\begin{aligned} X_1^r &= \{2\lambda : 2\lambda_{2r-1} < B_{2r-1}^-\}, \\ X_2^r &= \{2\lambda : 2\lambda_{2r-1} \geq B_{2r-1}^-\}, \end{aligned}$$

and for  $1 \leq s \leq \frac{n_0}{2} + 1$ ,

$$\begin{aligned} Y_1^s &= \{2\lambda : 2\lambda_{2j} \leq B_{2j}^+\}, \\ Y_2^s &= \{2\lambda : 2\lambda_{2j} > B_{2j}^+\}. \end{aligned}$$

Now we let

$$S(n) = \{(i, j) : 0 \leq i \leq \frac{n_1+1}{2}, 0 \leq j \leq \frac{n_0}{2}\}. \quad (5.2.1)$$

Then for  $(i, j) \in S(n)$ , we form the intersection

$$\begin{aligned} L_{i,j}(n, \sigma, \alpha) = & (X_2^1 \cap \cdots \cap X_2^i \cap X_1^{i+1} \cap \cdots \cap X_1^{\frac{n_1+1}{2}+1}) \cap \\ & (Y_2^1 \cap \cdots \cap Y_2^j \cap Y_1^{j+1} \cap \cdots \cap Y_1^{\frac{n_0}{2}+1}). \end{aligned}$$

More precisely,  $2\lambda \in L_{i,j}(n, \sigma, \alpha)$  if and only if

$$2\lambda_{2i-1} \geq B_{2i-1}^- \geq 2\lambda_{2i+1} \quad \text{and} \quad 2\lambda_{2j} \geq B_{2j+2}^+ \geq 2\lambda_{2j+2}, \quad (5.2.2)$$

where

$$B_{2i-1}^- = \sigma - \frac{n+\alpha+1}{2} + 2i - 1 + 1 = \sigma - \frac{n+\alpha}{2} + 2i - \frac{1}{2}$$

and

$$B_{2j+2}^+ = -\sigma - \frac{n+\alpha+1}{2} + 2j + 2 - 1 = -\sigma - \frac{n+\alpha}{2} + 2j + \frac{1}{2}.$$

Note that  $L_{a,b}(n, \sigma, \alpha)$  may be empty. If it is nonempty, then we shall identify it with the direct sum of all the  $K$  representations  $V_{2\lambda+\frac{\alpha}{2}\mathbf{1}}$  with  $2\lambda \in L_{a,b}(n, \sigma, \alpha)$ .

**Remark.** Let  $k = [n/2]$ . Then

$$S(n) = \begin{cases} \{(i, j) : 0 \leq i \leq k+1, 0 \leq j \leq k\} & n \text{ odd,} \\ \{(i, j) : 0 \leq i, j \leq k\} & n \text{ even.} \end{cases} \quad (5.2.3)$$

The following combinatorial result is elementary.



- Lemma 5.2.4.** (i) Assume that  $(i, j) \in S(n)$  and  $i - j \leq -1$ . Then  $L_{i,j} \neq \emptyset$  if and only if  $i - j \geq -\sigma + \frac{1}{2}$ . In particular, if  $\sigma \leq -1/2$ , then all such  $L_{i,j}$  is empty.
- (ii) For  $0 \leq i \leq k$ ,  $L_{i,i}$  is always nonempty.
- (iii) If  $(j+1, j) \in S(n)$ , then  $L_{j+1,j}$  is always nonempty.
- (iv) Assume that  $(i, j) \in S(n)$  and  $i - j \geq 2$ . Then  $L_{i,j} \neq \emptyset$  if and only if  $i - j \leq -\sigma + \frac{1}{2}$ . In particular, if  $\sigma \geq 1/2$ , then all such  $L_{i,j}$  is empty.

**Theorem 5.2.5.** (Case 2a) Assume that  $n + \alpha$  is even,  $\sigma + \frac{1}{2} \in \mathbb{Z}$ , and  $\tilde{\sigma} = \sigma + \frac{n+\alpha+1}{2}$  is odd.

- (i) If  $\sigma \geq 1/2$ , then

$$I^\alpha(\sigma) = \bigoplus \{L_{i,j}(n, \sigma, \alpha) : (i, j) \in S(n), -1 \leq j - i \leq r_1\},$$

where

$$r_1 = \min\left(\sigma - \frac{1}{2}, \left[\frac{n}{2}\right]\right),$$

and each  $L_{i,j}(n, \sigma, \alpha)$  which appears in the sum forms an irreducible constituent of  $I^\alpha(\sigma)$ . The module diagram of  $I^\alpha(\sigma)$  can be obtained from Figure 3 by removing those  $L_{a,b}(n, \sigma, \alpha)$  which are empty. In particular, the socle series of  $I^\alpha(\sigma)$  is given by

$$\text{Soc}^l(I^\alpha(\sigma)) = \begin{cases} \bigoplus_{(i,j) \in S(n), -1 \leq j-i \leq l-2} L_{i,j}(n, \sigma, \alpha) & 1 \leq l \leq r_1 + 1, \\ I^\alpha(\sigma) & l \geq r_1 + 2. \end{cases}$$

An irreducible constituent  $L_{i,j}(n, \sigma, \alpha)$  of  $I^\alpha(\sigma)$  is unitarizable if and only if  $i = j + 1$  or  $\frac{1}{2} \leq \sigma \leq \left[\frac{n}{2}\right] + \frac{1}{2}$  and  $j - i = r_1$ .

- (ii) If  $\sigma \leq -1/2$ , then

$$I^\alpha(\sigma) = \bigoplus \{L_{i,j}(n, \sigma, \alpha) : (i, j) \in S(n), 0 \leq i - j \leq r_2\},$$

where

$$r_2 = \min\left(-\sigma + \frac{1}{2}, \left[\frac{n+1}{2}\right]\right),$$

and each  $L_{i,j}(n, \sigma, \alpha)$  which appears in the sum forms an irreducible constituent of  $I^\alpha(\sigma)$ . The module diagram of  $I^\alpha(\sigma)$  can be obtained from Figure 3 by removing those  $L_{a,b}(n, \sigma, \alpha)$  which are empty. In particular, the socle series of  $I^\alpha(\sigma)$  is given by

$$\text{Soc}^l(I^\alpha(\sigma)) = \begin{cases} \bigoplus_{(i,j) \in S(n), r_2-l+1 \leq i-j \leq r_2} L_{i,j}(n, \sigma, \alpha) & 1 \leq l \leq r_2, \\ I^\alpha(\sigma) & l \geq r_2 + 1. \end{cases}$$

An irreducible constituent  $L_{i,j}(n, \sigma, \alpha)$  of  $I^\alpha(\sigma)$  is unitarizable if and only if  $i = j$  or  $-\left[\frac{n+1}{2}\right] + \frac{1}{2} \leq \sigma \leq -\frac{1}{2}$  and  $i - j = r_2$ .

The module diagram of  $I^\alpha(\sigma)$  in the case when  $n + \alpha$  is even and  $\sigma + \frac{n+\alpha+1}{2}$  is odd can be obtained from the following rectangle (Figure 3) by removing those  $L_{a,b}(n, \sigma, \alpha)$  which are empty. Here  $k = \lfloor n/2 \rfloor$  and we write a constituent  $L_{a,b}(n, \sigma, \alpha)$  simply as  $L_{a,b}$ . Note also that when  $n$  is even, the spaces  $L_{k+1,j}$  ( $0 \leq j \leq k$ ) are not defined. In this case, the rectangle below reduces to a square.

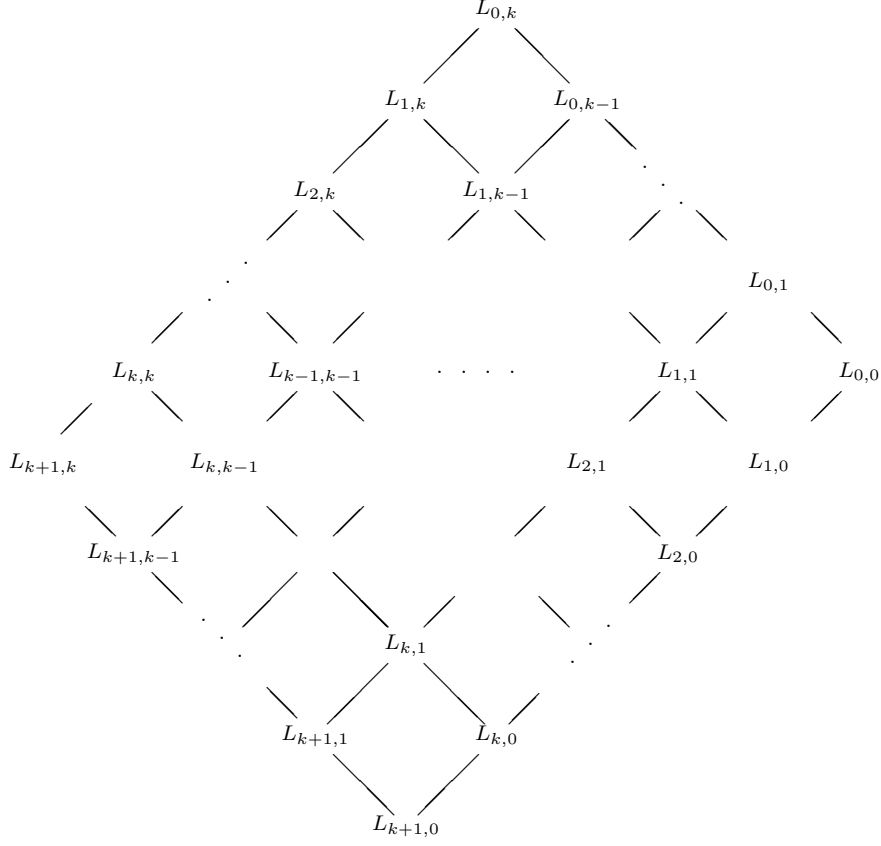


Figure 3: Diagram for  $I^\alpha(\sigma)$  when  $n + \alpha$  is even and  $\sigma + \frac{n+\alpha+1}{2}$  is odd

**Case 2b:**  $\sigma + \frac{n+\alpha+1}{2}$  is even

As in Case 2a, exactly one barrier is effective along each coordinate axis. But now  $\ell_j^+$  is effective when  $j$  is odd, and  $\ell_j^-$  is effective when  $j$  is even. So we have to define  $L_{a,b}(n, \sigma, \alpha)$  differently. For  $1 \leq r \leq \frac{n_1+1}{2}+1$ , let

$$\begin{aligned} X_1^r &= \{2\lambda : 2\lambda_{2r-1} \leq B_{2r-1}^+\}, \\ X_2^r &= \{2\lambda : 2\lambda_{2r-1} > B_{2r-1}^-\}, \end{aligned}$$

and for  $1 \leq s \leq \frac{n_0}{2} + 1$ ,

$$\begin{aligned} Y_1^s &= \{2\lambda : 2\lambda_{2s} < B_{2s}^-\}, \\ Y_2^s &= \{2\lambda : 2\lambda_{2s} \geq B_{2s}^-\}. \end{aligned}$$

We define the set  $S(n)$ , as in Case 2a in equation (5.2.1). Then for  $(i, j) \in S(n)$ , we form the intersection

$$\begin{aligned} L_{i,j}(n, \sigma, \alpha) = & (X_2^1 \cap \cdots \cap X_2^i \cap X_1^{i+1} \cap \cdots \cap X_1^{\frac{n_1+1}{2}+1}) \cap \\ & (Y_2^1 \cap \cdots \cap Y_2^j \cap Y_1^{j+1} \cap \cdots \cap Y_1^{\frac{n_0}{2}+1}). \end{aligned}$$

More precisely,  $2\lambda \in L_{i,j}(\sigma, \alpha)$  if and only if

$$2\lambda_{2i-1} \geq B_{2i+1}^+ \geq 2\lambda_{2i+1} \quad \text{and} \quad 2\lambda_{2j} \geq B_{2j}^- \geq 2\lambda_{2j+2}. \quad (5.2.6)$$

where

$$B_{2i+1}^+ = -\sigma - \frac{n + \alpha + 1}{2} + 2i + 1 - 1 = -\sigma - \frac{n + \alpha}{2} + 2i - \frac{1}{2}$$

and

$$B_{2j}^- = \sigma - \frac{n + \alpha + 1}{2} + 2j + 1 = \sigma - \frac{n + \alpha}{2} + 2j + \frac{1}{2}.$$

Note that  $L_{a,b}(n, \sigma, \alpha)$  may be empty. If it is nonempty, then we shall identify it with the direct sum of all the  $K$ -representations  $V_{2\lambda + \frac{\alpha}{2}\mathbf{1}}$  with  $2\lambda \in L_{a,b}(n, \sigma, \alpha)$ .

**Theorem 5.2.7.** (Case 2b) *Assume that  $n + \alpha$  is even,  $\sigma + \frac{1}{2} \in \mathbb{Z}$ , and  $\tilde{\sigma} = \sigma + \frac{n + \alpha + 1}{2}$  is even.*

(i) *If  $\sigma \geq 1/2$ , then*

$$I^\alpha(\sigma) = \bigoplus \{L_{i,j}(n, \sigma, \alpha) : (i, j) \in S(n), 0 \leq i - j \leq r_2\},$$

where

$$r_2 = \min(\sigma + \frac{1}{2}, \lfloor \frac{n+1}{2} \rfloor),$$

and each  $L_{i,j}(n, \sigma, \alpha)$  which appears in the sum forms an irreducible constituent of  $I^\alpha(\sigma)$ . The module diagram of  $I^\alpha(\sigma)$  can be obtained from Figure 4 by removing those  $L_{a,b}(n, \sigma, \alpha)$  which are empty. In particular, the socle series of  $I^\alpha(\sigma)$  is given by

$$\text{Soc}^l(I^\alpha(\sigma)) = \begin{cases} \bigoplus_{(i,j) \in S(n), 0 \leq i-j \leq l-1} L_{i,j}(n, \sigma, \alpha) & 1 \leq l \leq r_2, \\ I^\alpha(\sigma) & l \geq r_2 + 1. \end{cases}$$

An irreducible constituent  $L_{i,j}(n, \sigma, \alpha)$  of  $I^\alpha(\sigma)$  is unitarizable if and only if  $i = j$  or  $\frac{1}{2} \leq \sigma \leq \lfloor \frac{n+1}{2} \rfloor - \frac{1}{2}$  and  $i - j = r_2$ .

(ii) If  $\sigma \leq -1/2$ , then

$$I^\alpha(\sigma) = \bigoplus \{L_{i,j}(n, \sigma, \alpha) : (i, j) \in S(n), -1 \leq j - i \leq r_1\},$$

where

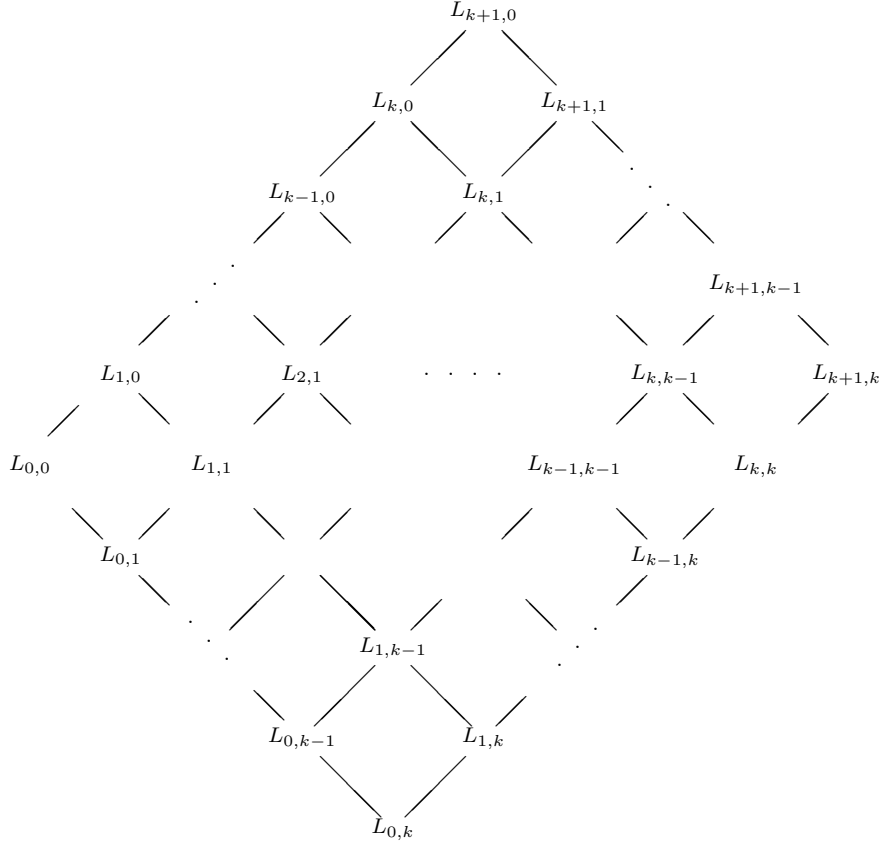
$$r_1 = \min \left( -\sigma - \frac{1}{2}, \left[ \frac{n}{2} \right] \right),$$

and each  $L_{i,j}(n, \sigma, \alpha)$  which appears in the sum forms an irreducible constituent of  $I^\alpha(\sigma)$ . The module diagram of  $I^\alpha(\sigma)$  can be obtained from Figure 4 by removing those  $L_{a,b}(n, \sigma, \alpha)$  which are empty. In particular, the socle series of  $I^\alpha(\sigma)$  is given by

$$\text{Soc}^l(I^\alpha(\sigma)) = \begin{cases} \bigoplus_{(i,j) \in S(n), r_1-l+1 \leq j-i \leq r_1} L_{i,j}(n, \sigma, \alpha) & 1 \leq l \leq r_1 + 1, \\ I^\alpha(\sigma) & l \geq r_1 + 2. \end{cases}$$

An irreducible constituent  $L_{i,j}(n, \sigma, \alpha)$  of  $I^\alpha(\sigma)$  is unitarizable if and only if  $i = j + 1$  or  $-\left[\frac{n}{2}\right] - \frac{1}{2} \leq \sigma \leq -\frac{1}{2}$  and  $j - i = r_1$ .

Finally, we describe the module diagram of  $I^\alpha(\sigma)$  in the case when both  $n + \alpha$  and  $\sigma + \frac{n + \alpha + 1}{2}$  are even. It can be obtained from the following rectangle (Figure 4) by removing those  $L_{a,b}(n, \sigma, \alpha)$  which are empty. As before,  $k = [n/2]$ , and we write a constituent  $L_{a,b}(n, \sigma, \alpha)$  simply as  $L_{a,b}$ . When  $n$  is even,  $L_{k+1,j}$  ( $0 \leq j \leq k$ ) is empty so that this rectangle reduces to a square.

Figure 4: Diagram for  $I^\alpha(\sigma)$  when both  $n + \alpha$  and  $\sigma + \frac{n+\alpha+1}{2}$  are even

## 6. THE IMAGE OF $\psi_{p,q}$ IN $I^\alpha(\sigma)$

In this section, we will describe the image of  $\psi_{p,q}$  in  $I^\alpha(\sigma)$ :

$$\psi_{p,q} : \Omega^{p,q} \hookrightarrow I^\alpha(\sigma),$$

where

$$\sigma = \frac{p+q}{2} - \frac{n+1}{2}, \text{ and } \alpha \equiv p-q \pmod{4}.$$

We allow  $(p,q) = (0,0)$ , in which case we understand  $\Omega^{0,0}$  to be the trivial representation of  $G = \widetilde{\text{Sp}}(2n, \mathbb{R})$ .

Denote  $m = p + q$ . Observe that

$$n + \alpha \equiv n + m \pmod{2},$$

and

$$\tilde{\sigma} = \sigma + \frac{n + \alpha + 1}{2} = \frac{m + \alpha}{2} \equiv p \pmod{2}.$$

Recall that our analysis of  $I^\alpha(\sigma)$  is divided into 4 cases:

$\tilde{\sigma} \setminus n + \alpha$	odd	even
odd	Case 1a	Case 2a
even	Case 1b	Case 2b

Correspondingly, our analysis of  $\Omega^{p,q}$  will also be divided into 4 cases:

$p \setminus n + m$	odd	even
odd	Case 1a	Case 2a
even	Case 1b	Case 2b

Throughout this section, we assume that we are at a point of reducibility, namely

$$\sigma + \frac{n + \alpha + 1}{2} \in \mathbb{Z}.$$

If  $\frac{p+q-(n+1)}{2} = \sigma$  and  $p - q \equiv \alpha \pmod{4}$ , we say that we have a possible embedding of  $\Omega^{p,q}$  (into  $I^\alpha(\sigma)$ ).

We shall give the detailed description of  $\Omega^{p,q}$  in  $I^\alpha(\sigma)$  in the following two subsections. We end this section with a corollary which gives conceptual underpinning to our results.

**Theorem 6.1.** *The relationship between  $\Omega^{p,q}$ 's and  $I^\alpha(\sigma)$  is as follows.*

- (a)  $-\rho_n \leq \sigma < 0$ : *The irreducible submodules of  $I^\alpha(\sigma)$  are given by the possible embeddings of  $\Omega^{p,q}$ 's, and all of them are unitary.*
- (b) *Unitary axis ( $\sigma = 0$ ) (when  $n + \alpha$  is odd):*

$$I^\alpha(0) = \bigoplus_{\substack{p+q=n+1 \\ p-q \equiv \alpha \pmod{4}}} \Omega^{p,q}.$$

- (c)  $\sigma > 0$ : *The reducibilities of  $I^\alpha(\sigma)$  are completely accounted for by the possible embeddings of  $\Omega^{p,q}$ 's.*
  - *See Part b) of Theorems 6.1.2, 6.1.3, 6.2.2 and 6.2.3 for the precise statements.*

### 6.1. The image of $\psi_{p,q}$ in $I^\alpha(\sigma)$ : $m$ and $n$ different parity.

#### Case 1: $n + m$ odd.

We consider two subcases:

$$\text{Case 1a: } p \text{ is odd} \qquad \text{Case 1b: } p \text{ is even.}$$

**Case 1a:** we write

$$p = 2i + 1, \quad q = 2j + \epsilon,$$

so that  $p + q = m$ . Here

$$\epsilon = \begin{cases} 1 & m \text{ even,} \\ 0 & m \text{ odd.} \end{cases}$$

Note that

$$i + j = \frac{m - 1 - \epsilon}{2} = \frac{m + n_0 - (n + 1)}{2} = \frac{n_0}{2} + \sigma.$$

**Case 1b:** we write

$$p = 2i, \quad q = 2j + \epsilon',$$

so that  $p + q = m$ . Here

$$\epsilon' = \begin{cases} 0 & m \text{ even,} \\ 1 & m \text{ odd.} \end{cases}$$

Note that

$$i + j = \frac{m - \epsilon'}{2} = \frac{m + (n_1 + 1) - (n + 1)}{2} = \frac{n_1 + 1}{2} + \sigma.$$

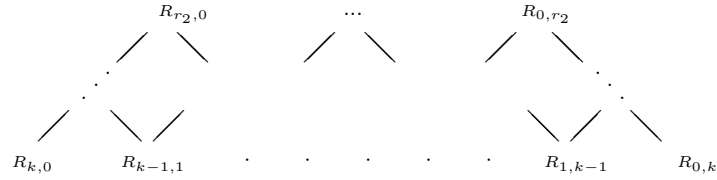
In accordance with 5.1.13, we let

$$k = \begin{cases} \frac{n_0}{2} = [\frac{n}{2}] & \text{if } p \text{ is odd,} \\ \frac{n_1 + 1}{2} = [\frac{n + 1}{2}] & \text{if } p \text{ is even.} \end{cases}$$

Recall that we have the following module diagrams:



Case 1: Module diagram for  $I^\alpha(\sigma)$  ( $-k \leq \sigma \leq 0$ );  $r_1 = k + \sigma$



Case 1: Module diagram for  $I^\alpha(\sigma)$  ( $1 \leq \sigma \leq k$ );  $r_2 = k - \sigma$

We introduce one notation. For an irreducible submodule  $R_{s,t}$  of  $I^\alpha(\sigma)$  where  $\sigma > 0$ , denote by  $\prec R_{s,t} \succ$  the submodule of  $I^\alpha(\sigma)$  generated by  $R_{s,t}$ :

$$\prec R_{s,t} \succ = \oplus \{R_{i,j} : i \geq s, j \geq t\}. \quad (6.1.1)$$

Recall that

$$\sigma = \frac{m - (n + 1)}{2}, \quad \text{and} \quad p - q \equiv \alpha \pmod{4}.$$

**Theorem 6.1.2.** (Case 1a) *Assume that  $n + m$  is odd, and  $p$  is odd.*

(a) *If  $\sigma \leq 0$ , then*

$$\Omega^{p,q} = R_{\frac{p-1}{2}, \frac{q-\epsilon}{2}}.$$

(b) *If  $\sigma \geq 1$ , then*

$$\Omega^{p,q} = \prec R_{s,t} \succ,$$

*where  $s = \max(0, \frac{n-q}{2})$ , and  $t = \max(0, \frac{n+1-\epsilon-p}{2})$ .*

**Remarks:**

- (a) For  $(i, j) = (\frac{p-1}{2}, \frac{q-\epsilon}{2})$ , we have noted that  $i + j = \frac{n_0}{2} + \sigma = r_1$ . Thus  $R_{i,j}$  is at the bottom layer of the module diagram of  $I^\alpha(\sigma)$ , namely is an irreducible submodule. The collection of  $R_{i,j}$ 's exhausts the set of all irreducible submodules of  $I^\alpha(\sigma)$ .
- (b) If  $q \leq n$ , and  $p \leq n + 1 - \epsilon$ , then  $s = \max(0, \frac{n-q}{2}) = \frac{n-q}{2}$ ,  $t = \max(0, \frac{n+1-\epsilon-p}{2}) = \frac{n+1-\epsilon-p}{2}$ , and  $s + t = \frac{n-\epsilon}{2} - \sigma = \frac{n_0}{2} - \sigma = r_2$ . Such  $R_{s,t}$ 's are exactly those at the top layer of the module diagram of  $I^\alpha(\sigma)$ , namely irreducible quotients. The rest of  $R_{s,t}$ 's are those on the “left boundary” ( $s = 0$ ) or the “right boundary” ( $t = 0$ ).

**Theorem 6.1.3.** (Case 1b) *Assume that  $n + m$  is odd, and  $p$  is even.*

(a) *If  $\sigma \leq 0$ , then*

$$\Omega^{p,q} = R_{\frac{p}{2}, \frac{q-\epsilon'}{2}}.$$

(b) *If  $\sigma \geq 1$ , then*

$$\Omega^{p,q} = \prec R_{s,t} \succ,$$

*where  $s = \max(0, \frac{n+1-q}{2})$ , and  $t = \max(0, \frac{n+1-\epsilon'-p}{2})$ .*



**Remarks:**

- (a) For  $(i, j) = (\frac{p}{2}, \frac{q-\epsilon'}{2})$ , we have noted that  $i + j = \frac{n+1}{2} + \sigma = r_1$ . Thus  $R_{i,j}$  is at the bottom layer of the module diagram of  $I^\alpha(\sigma)$ , namely is an irreducible submodule. The collection of  $R_{i,j}$ 's exhausts the set of all irreducible submodules of  $I^\alpha(\sigma)$ .
- (b) If  $q \leq n+1$ , and  $p \leq n+1-\epsilon'$ , then  $s = \max(0, \frac{n+1-q}{2}) = \frac{n+1-q}{2}$ ,  $t = \max(0, \frac{n+1-\epsilon'-p}{2}) = \frac{n+1-\epsilon'-p}{2}$ , and  $s + t = \frac{n+1-\epsilon'}{2} - \sigma = \frac{n+1}{2} - \sigma = r_2$ . Such  $R_{s,t}$ 's are exactly those at the top layer of the module diagram of  $I^\alpha(\sigma)$ , namely irreducible quotients. The rest of  $R_{s,t}$ 's are those on the “left boundary” ( $s = 0$ ) or the “right boundary” ( $t = 0$ ).

**6.2. The image of  $\psi_{p,q}$  in  $I^\alpha(\sigma)$ :  $m$  and  $n$  same parity.****Case 2:  $n + m$  even.**

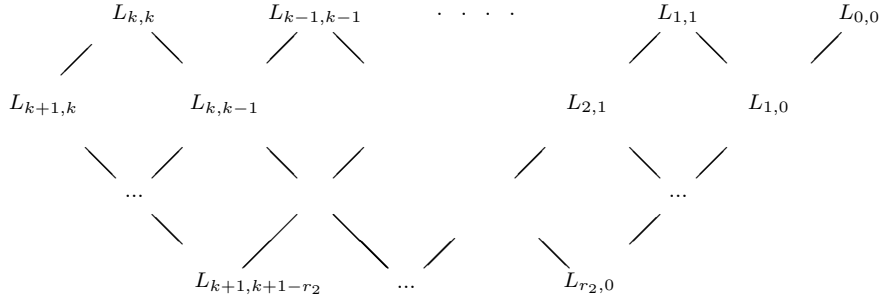
We consider two subcases:

*Case 2a:  $p$  is odd*      *Case 2b:  $p$  is even.*

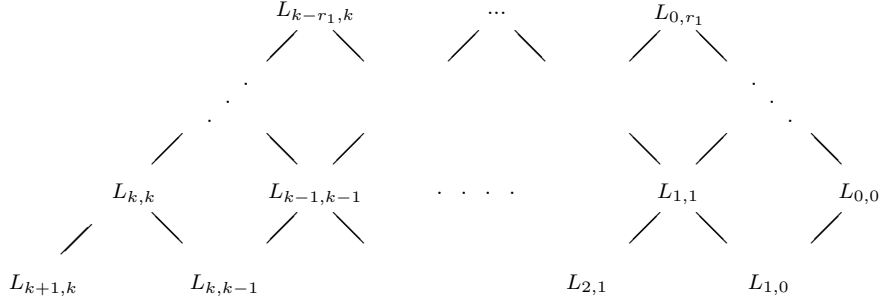
We introduce one similar notation in this case. For an irreducible submodule  $L_{s,t}$  of  $I^\alpha(\sigma)$  where  $\sigma > 0$ , denote by  $\prec L_{s,t} \succ$  the submodule of  $I^\alpha(\sigma)$  generated by  $L_{s,t}$ :

$$\prec L_{s,t} \succ = \begin{cases} \oplus \{L_{i,j} : i \geq s, j \leq t\}, & \text{(Case 2a),} \\ \oplus \{L_{i,j} : i \leq s, j \geq t\}, & \text{(Case 2b).} \end{cases} \quad (6.2.1)$$

As in Subsection 5.2, we let  $k = \lfloor n/2 \rfloor$ . First recall the module diagram of  $I^\alpha(\sigma)$  for Case 2a:



Case 2a: Diagram for  $I^\alpha(\sigma)$  when  $n + m$  is even and  $p$  is odd;  $\sigma \leq -\frac{1}{2}$ ;  $r_2 = -\sigma + \frac{1}{2} \leq \lfloor \frac{n+1}{2} \rfloor$



Case 2a: Diagram for  $I^\alpha(\sigma)$  when  $n+m$  is even and  $p$  is odd;  $\sigma \geq \frac{1}{2}$ ;  $r_1 = \sigma - \frac{1}{2} \leq [\frac{n}{2}]$

**Theorem 6.2.2.** (Case 2a) *Assume that  $n+m$  is even, and  $p$  is odd.*

(a) *If  $\sigma \leq -\frac{1}{2}$ , then*

$$\Omega^{p,q} = L_{\frac{n+1-q}{2}, \frac{p-1}{2}}.$$

(b) *If  $\sigma \geq \frac{1}{2}$ , then*

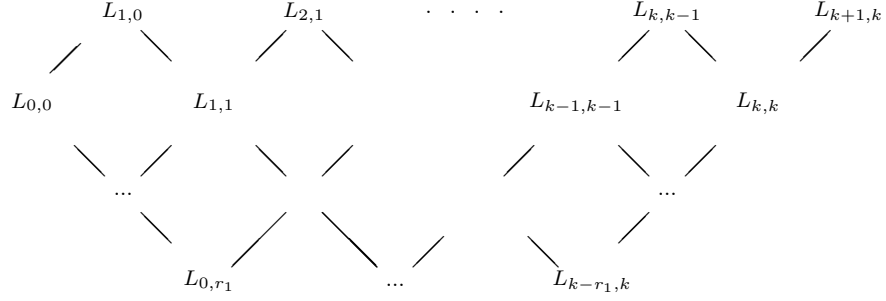
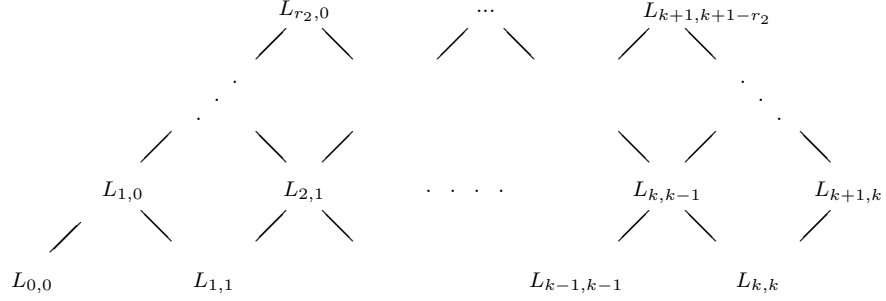
$$\Omega^{p,q} = \prec L_{s,t} \succ,$$

*where  $s = \max(0, \frac{n+1-q}{2})$ , and  $t = \min([\frac{n}{2}], \frac{p-1}{2})$ .*

**Remarks:**

- (a) For  $(i, j) = (\frac{n+1-q}{2}, \frac{p-1}{2})$ , we have  $i - j = \frac{n+2-m}{2} = -\sigma + \frac{1}{2} = r_2$ . Thus  $L_{i,j}$  is at the bottom layer of the module diagram of  $I^\alpha(\sigma)$ , namely is an irreducible submodule. The collection of  $L_{i,j}$ 's exhausts the set of all irreducible submodules of  $I^\alpha(\sigma)$ .
- (b) If  $q \leq n+1$ , and  $\frac{p-1}{2} \leq [\frac{n}{2}]$ , then  $s = \max(0, \frac{n+1-q}{2}) = \frac{n+1-q}{2}$ ,  $t = \min([\frac{n}{2}], \frac{p-1}{2}) = \frac{p-1}{2}$ , and  $t-s = \frac{m-(n+2)}{2} = \sigma - \frac{1}{2} = r_1$ . Such  $L_{s,t}$ 's are exactly those at the top layer of the module diagram of  $I^\alpha(\sigma)$ , namely irreducible quotients. The rest of  $L_{s,t}$ 's are those on the “left boundary” ( $t = k = [\frac{n}{2}]$ ) or the “right boundary” ( $s = 0$ ). Note that when  $n$  is even, the submodule  $L_{k+1,k}$  is empty.

Next recall the module diagram of  $I^\alpha(\sigma)$  for Case 2b:


 Case 2b: Diagram for  $I^\alpha(\sigma)$  when  $n+m$  and  $p$  are even;  $\sigma \leq -\frac{1}{2}$ ;  $r_1 = -\sigma - \frac{1}{2} \leq [\frac{n}{2}]$ 

 Case 2b: Diagram for  $I^\alpha(\sigma)$  when both  $n+m$  and  $p$  are even;  $\sigma \geq \frac{1}{2}$ ;  $r_2 = \sigma + \frac{1}{2} \leq [\frac{n+1}{2}]$ 

**Theorem 6.2.3.** (Case 2b) *Assume that  $n+m$  is even, and  $p$  is even.*

(a) *If  $\sigma \leq -\frac{1}{2}$ , then*

$$\Omega^{p,q} = L_{\frac{p}{2}, \frac{n-q}{2}}.$$

(b) *If  $\sigma \geq \frac{1}{2}$ , then*

$$\Omega^{p,q} = \prec L_{s,t} \succ,$$

*where  $s = \min([\frac{n+1}{2}], \frac{p}{2})$ , and  $t = \max(0, \frac{n-q}{2})$ .*

**Remarks:**

- (a) For  $(i, j) = (\frac{p}{2}, \frac{n-q}{2})$ , we have  $j-i = \frac{n-m}{2} = -\sigma - \frac{1}{2} = r_1$ . Thus  $L_{i,j}$  is at the bottom layer of the module diagram of  $I^\alpha(\sigma)$ , namely is an irreducible submodule. The collection of  $L_{i,j}$ 's exhausts the set of all irreducible submodules of  $I^\alpha(\sigma)$ .

- (b) If  $\frac{p}{2} \leq [\frac{n+1}{2}]$  and  $q \leq n$ , then  $s = \min([\frac{n+1}{2}], \frac{p}{2}) = \frac{p}{2}$ ,  $t = \max(0, \frac{n-q}{2}) = \frac{n-q}{2}$ , and  $s - t = \frac{m-n}{2} = \sigma + \frac{1}{2} = r_2$ . Such  $L_{s,t}$ 's are exactly those at the top layer of the module diagram of  $I^\alpha(\sigma)$ , namely irreducible quotients. The rest of  $L_{s,t}$ 's are those on the “left boundary” ( $t = 0$ ) or the “right boundary” ( $s = [\frac{n+1}{2}]$ ). Note that when  $n$  is even, the subquotient  $L_{k+1,j}$  is empty.

## REFERENCES

- [H1] R. Howe,  *$\theta$ -series and invariant theory*, in Automorphic Forms, Representations and  $L$ -functions, Proc. Symp. Pure Math. 33, (1979), 275–285.
- [H2] R. Howe, *Transcending classical invariant theory*, J. Amer. Math. Soc. 2, (1989), 535–552.
- [Ku] S. S. Kudla, *Notes on the local theta correspondence*, Lecture notes from the European School of Group Theory, 1996. <http://www.math.toronto.edu/~skudla/ssk.research.html>.
- [KR1] S. S. Kudla and S. Rallis, *Degenerate principal series and invariant distributions*, Israel J. Math. 69, (1990), 25–45.
- [KR2] S. S. Kudla and S. Rallis, *Ramified degenerate principal series for  $Sp(n)$* , Israel J. Math. 78, (1992), 209–256.
- [KR3] S. S. Kudla and S. Rallis, *A regularized Siegel-Weil formula: The first term identity*, Annals Math. 140, (1994), 1–80.
- [KS] S. S. Kudla and W. J. Sweet, Jr., *Degenerate principal series representations for  $U(n, n)$* , Israel J. Math. 98, (1997), 253–306.
- [L1] S. T. Lee, *On some degenerate principal series representations for  $U(n, n)$* , J. Funct. Anal. 126, (1994), 305–366.
- [L2] S. T. Lee, *Degenerate principal series representations for  $Sp(2n, \mathbb{R})$* , Compositio Math. 103, (1996), 123–151.
- [LZ1] S. T. Lee and C.-B. Zhu, *Degenerate principal series and local theta correspondence*, Trans. Amer. Math. Soc. 350, (1998), 5017–5046.
- [LZ2] S. T. Lee and C.-B. Zhu, *Degenerate principal series and local theta correspondence II*, Israel J. Math. 100, (1997), 29–59.
- [LZ3] S. T. Lee and C.-B. Zhu, *Degenerate principal series and local theta correspondence III: the case of complex groups*, J. Algebra 319, (2008), 336–359.
- [Ra] S. Rallis, *On the Howe duality conjecture*, Compositio Math. 51, (1984), 333–399.
- [SZ] B. Sun and C.-B. Zhu, *Conservation relations for local theta correspondence*, arXiv:1204.2969v1.
- [Ya] S. Yamana, *Degenerate principal series representations for quaternionic unitary groups*, Israel J. Math. 185, (2011), 77–124.
- [Zh] C.-B. Zhu, *Invariant distributions of classical groups*, Duke Math. J. 65, (1992), 85–119.

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